Primitive Exponent of a Class of Special Three-Colored Digraph

Meijin LUO [1], Yuanhua LIN [1,*], Xi Li [2]

[1]School of Mathematics and Statistics, Hechi University, Yizhou, Guangxi 546300, P.R. China
[2]Department of Basic Education, Shanxi Yuncheng Vocational College of Agriculture, Yuncheng, Shanxi 044000, P.R. China
*Corresponding author: Yuanhua LIN

Abstract: According to the corresponding relationship between matrix and digraph, the upper bound of primitive exponent of a class of special three-colored digraph whose uncolored digraph has $n-1$ vertices, $n+1$ arcs and consists of three cycles is studied. With the help of the inverse of the cycle matrix, we obtain the upper bound of the primitive exponent. Thus, the problem of the upper bound of primitive exponent of a class of corresponding nonnegative Matrix tuple is solved.

Keywords: upper bound; three-colored; digraph; exponent

1. Introduction

A three-colored digraph D is a strongly connected digraph whose arcs are divided by red, blue and yellow. Given a walk ω in D, $r(\omega)$, $s(\omega)$ and $t(\omega)$ represent respectively the number of red arcs, blue arcs and yellow arcs of ω, and the composition of ω is the vector $(r(\omega),s(\omega),t(\omega))$ or $(r(\omega),s(\omega),t(\omega))^T$ [1].

A three-colored digraph D is primitive if and only if there exist nonnegative integers h, k and v with $h+k+v>0$ such that for each pair (i,j) of vertices there is an $(h,k,v)-$ walk in D from i to j. The primitive exponent $\exp(D)$, is defined to be the smallest value of $h+k+v$ over all such h, k and v [2].

Let $C=\{\gamma_1,\gamma_2,\cdots,\gamma_l\}$ be the set of cycles of D. We can assume that M is the cycle matrix of D, and

$$M=\begin{bmatrix} a_1 & a_2 & \cdots & a_l \\ b_1 & b_2 & \cdots & b_l \\ c_1 & c_2 & \cdots & c_l \end{bmatrix}.$$

$a_i, b_i, c_i\,(i=1,2,\cdots,l)$ represent respectively the number of red arcs, blue arcs and yellow arcs in γ_i. The content (M), is defined to be 0 if the rank$(M)<3$ and the greatest common divisor of all 3×3 minors of M, otherwise [3].

Lemma 1 Let D be a three-colored digraph having at least one red arc, one blue arc and one yellow arc. Then D is primitive if and only if D is strongly connected and content $(M)=1$ [2].

In fact, using the simple and intuitive characteristics of graph theory, the related problems of the primitive exponent of nonnegative matrix pair can be transformed into two-colored digraph to solve [1-6]. Similarly, the three-colored digraph is used to solve the related problems of primitive exponents of corresponding of nonnegative matrix tuple. So far, the research on the primitive exponent of nonnegative matrix tuple has only obtained some results [7-10]. At present, by using the corresponding relationship between matrix and graph, the research of three-colored digraph has specific application background in many aspects such as communication network, computer science, economics and so on. For example, it plays an important role to find the strong edge coloring of bipartite graph in coding cache.

In this paper, for any $n\geq 4 (n\in Z^+)$, we consider a class of three-colored digraph D whose uncolored digraph is shown in Figure 1.

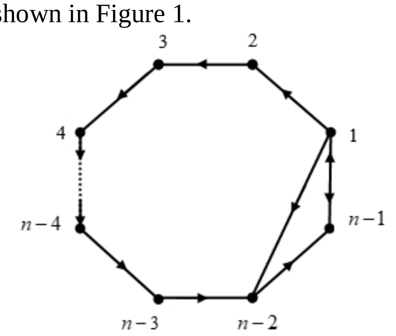

Fig 1. Uncolored digraph of D

Clearly, D contains $n-1$ vertices, $n+1$ arcs, one $(n-1)-$ cycle, one $3-$ cycle and one $2-$ cycle. From Lemma 1 and Fig 1, we know that all of arcs in $3-$cycle or $2-$cycle have the same color, that is, the composition of ω is the vector $(3,0,0)$, $(0,3,0)$ or $(0,0,3)$ in $3-$cycle, or the composition of ω is the vector $(2,0,0)$, $(0,2,0)$ or $(0,0,2)$ in $2-$cycle, then content $(M)\neq 1$ and D is non-primitive. So, all of arcs must have the different color in $3-$cycle and $2-$cycle. Without loss of generality, we may assume that the cycle matrix of D is

$$M=\begin{bmatrix} x & 1 & 1 \\ y & 1 & 1 \\ n-1-x-y & 1 & 0 \end{bmatrix} \tag{1}$$

for some nonnegative integers x, y.

Theorem 1 D is primitive if and only if $-x+y=\pm 1$.

Proof From (1), we know $|M|=-x+y$. By Lemma 1, D is primitive if and only if content$(M)=1$, that is, $|M|=\pm 1$. Then the theorem follows.

2. Upper Bound of Primitive Exponents

Theorem 2 Let D be primitive and $|M|=-x+y=1$, then

$$\exp(D)\le n^2-\frac{3n}{2}.$$

Proof In this case, the inverse matrix of the cycle matrix M is

$$M^{-1}=\begin{bmatrix}-1 & 1 & 0\\ n-2y & -n+2y & 1\\ 3y-n & -3y+n+1 & -1\end{bmatrix}.$$

Clearly, if $|M|=-x+y=1$, then $x=y-1$. Associating to the cycle matrix of M, we can see $n-1-x-y=n-1-(y-1)-y=n-2y\ge 0$, so $y\le\frac{n}{2}$, $-n+2y\le 0$. The size of matrix elements $3y-n$ and $-3y+n+1$ are related to the value of y in M^{-1}. For any pair (i,j) of vertices of D, let P_{ij} be the shortest path from i to j, and denote $r(P_{ij})=r$, $b(P_{ij})=s$, $y(P_{ij})=t$. Therefore, combined with the different values of y, the following four aspects are discussed.

Case 1: $y=\frac{n}{3}$ ($n=3k$, $k\ge 2$ and $k\in Z^+$).

If $y=\frac{n}{3}$, then

$$M=\begin{bmatrix}\frac{n}{3}-1 & 1 & 1\\ \frac{n}{3} & 1 & 1\\ \frac{n}{3} & 1 & 0\end{bmatrix},\ M^{-1}=\begin{bmatrix}-1 & 1 & 0\\ \frac{n}{3} & -\frac{n}{3} & 1\\ 0 & 1 & -1\end{bmatrix}.$$

We only need to prove that any pair of vertices (i,j) has an $(\frac{2n^2}{9},\frac{2n^2+3n}{9},\frac{2n^2}{9})-$walk in D. We can get the following results,

$$\begin{bmatrix}r\\ s\\ t\end{bmatrix}+(\frac{n}{3}+r-s)\begin{bmatrix}\frac{n}{3}-1\\ \frac{n}{3}\\ \frac{n}{3}\end{bmatrix}+(\frac{n^2}{9}-\frac{n}{3}r+\frac{n}{3}s-t)\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$$

$$+(\frac{n}{3}-s+t)\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}=\begin{bmatrix}\frac{2n^2}{9}\\ \frac{2n^2+3n}{9}\\ \frac{2n^2}{9}\end{bmatrix}. \qquad (2)$$

Clearly, $r\le\frac{n}{3}-1$, $s\le\frac{n}{3}$, $t\le\frac{n}{3}$. Combined with Fig.1, if $s=\frac{n}{3}$, then $r\ge 0$, $t\ge 0$; if $r=\frac{n}{3}-1$, $t=\frac{n}{3}$, then $s\ge 0$. Hence $\frac{n}{3}+r-s\ge 0$, $\frac{n^2}{9}-\frac{n}{3}r+\frac{n}{3}s-t\ge 0$ and $\frac{n}{3}-s+t\ge 0$. By (2), we can see that the walk starts at vertex i, follows P_{ij} to vertex j, goes $\frac{n}{3}+r-s$ times around the $(n-1)-$cycle, $\frac{n^2}{9}-\frac{n}{3}r+\frac{n}{3}s-t$ times around the $3-$cycle, and $\frac{n}{3}-s+t$ times around the $2-$cycle is an $(\frac{2n^2}{9},\frac{2n^2+3n}{9},\frac{2n^2}{9})-$walk. So

$$\exp(D)\le\frac{2n^2}{9}+\frac{2n^2+3n}{9}+\frac{2n^2}{9}=\frac{2n^2+n}{3}.$$

Case 2: $y=\frac{n+1}{3}$ ($n=3k-1$, $k\ge 2$ and $k\in Z^+$).

If $y=\frac{n+1}{3}$, then

$$M=\begin{bmatrix}\frac{n-2}{3} & 1 & 1\\ \frac{n+1}{3} & 1 & 1\\ \frac{n-2}{3} & 1 & 0\end{bmatrix},\ M^{-1}=\begin{bmatrix}-1 & 1 & 0\\ \frac{n-2}{3} & -\frac{n-2}{3} & 1\\ 1 & 0 & -1\end{bmatrix}.$$

We only need to prove that any pair of vertices (i,j) has an $(\frac{2n^2+n-10}{9},\frac{2n^2+4n-7}{9},\frac{2n^2-2n-4}{9})-$walk. We can get the following results,

$$\begin{bmatrix}r\\ s\\ t\end{bmatrix}+(\frac{n+1}{3}+r-s)\begin{bmatrix}\frac{n-2}{3}\\ \frac{n+1}{3}\\ \frac{n-2}{3}\end{bmatrix}+(\frac{n^2-n-2}{9}-\frac{n-2}{3}r+\frac{n-2}{3}s-t)\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$$

$$+(\frac{n-2}{3}-r+t)\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}=\begin{bmatrix}\frac{2n^2+n-10}{9}\\ \frac{2n^2+4n-7}{9}\\ \frac{2n^2-2n-4}{9}\end{bmatrix}. \qquad (3)$$

Clearly, $r\le\frac{n-2}{3}$, $s\le\frac{n+1}{3}$, $t\le\frac{n-2}{3}$. Combined with Fig.1, if $s=\frac{n+1}{3}$, then $r\ge 0$, $t\ge 0$; if $r=\frac{n-2}{3}$, $t=\frac{n-2}{3}$, then $s\ge 0$. Hence $\frac{n+1}{3}+r-s\ge 0$, $\frac{n^2-n-2}{9}-\frac{n-2}{3}r+\frac{n-2}{3}s-t\ge 0$ and $\frac{n-2}{3}-r+t\ge 0$. By (3), we can see that the walk starts at vertex i, follows P_{ij} to vertex j, goes $\frac{n+1}{3}+r-s$ times around the $(n-1)-$ cycle, $\frac{n^2-n-2}{9}-\frac{n-2}{3}r+\frac{n-2}{3}s-t$ times around the $3-$ cycle, and $\frac{n-2}{3}-r+t$ times around the $2-$ cycle is an $(\frac{2n^2+n-10}{9},\frac{2n^2+4n-7}{9},\frac{2n^2-2n-4}{9})-$ walk. So

$$\exp(D)\le\frac{2n^2+n-10}{9}+\frac{2n^2+4n-7}{9}+\frac{2n^2-2n-4}{9}$$
$$=\frac{2n^2+n-7}{3}.$$

Case 3: $1\le y<\frac{n}{3}(y\in Z^+)$.

If $1\le y<\frac{n}{3}$, then

$$M=\begin{bmatrix} y-1 & 1 & 1\\ y & 1 & 1\\ n-2y & 1 & 0\end{bmatrix},\ M^{-1}=\begin{bmatrix} -1 & 1 & 0\\ n-2y & -n+2y & 1\\ 3y-n & -3y+n+1 & -1\end{bmatrix}.$$

We only need to prove that any pair of vertices (i,j) in D has an $(2ny-4y^2, 2ny+y-4y^2, 2ny-4y^2)-$ walk. We can get the following results,

$$\begin{bmatrix} r\\ s\\ t\end{bmatrix}+(y+r-s)\begin{bmatrix} y-1\\ y\\ n-2y\end{bmatrix}+(ny-2y^2-(n-2y)r+(n-2y)s-t)\begin{bmatrix}1\\1\\1\end{bmatrix}$$
$$+(ny+y-3y^2+(n-3y)r-(n-3y+1)s+t)\begin{bmatrix}1\\1\\0\end{bmatrix}$$
$$=\begin{bmatrix} 2ny-4y^2\\ 2ny+y-4y^2\\ 2ny-4y^2\end{bmatrix}. \qquad (4)$$

Clearly, $r\le y-1$, $s\le y$, $t\le n-2y$, $3y-n<0$, $-3y+n+1>0$. Combined with Fig.1, if $s=y$, then $r\ge 0$, $t\ge 0$; if $r=y-1$, $t=n-2y$, then $s\ge 0$. Hence $y+r-s\ge 0$, $ny-2y^2-(n-2y)r+(n-2y)s-t\ge 0$ and $ny+y-3y^2+(n-3y)r-(n-3y+1)s+t\ge 0$. By (4), we can see that the walk starts at vertex i, follows P_{ij} to vertex j, goes $y+r-s$ times around the $(n-1)-$ cycle, $ny-2y^2-(n-2y)r+(n-2y)s-t$ times around the $3-$ cycle, and $ny+y-3y^2+(n-3y)r-(n-3y+1)s+t$ times around the $2-$ cycle is an $(2ny-4y^2, 2ny+y-4y^2, 2ny-4y^2)-$walk. So

$$\exp(D)\le 2ny-4y^2+2ny+y-4y^2+2ny-4y^2$$
$$=6ny-12y^2+y.$$

Obviously, $\exp(D)$ is a function of y. Denote $f(y)=6ny-12y^2+y$, then $f'(y)=6n-24y+1$ and $f(y)$ has a maximum when $y=\frac{6n+1}{24}$. Since $1\le y<\frac{n}{3}$ and $y\in Z^+$, we have that

$$\exp(D)\le 6ny-12y^2+y\le f(\frac{n}{4})=\frac{3n^2+n}{4}.$$

Case 4: $\frac{n+1}{3}<y\le\frac{n}{2}(y\in Z^+)$.

If $\frac{n+1}{3}<y\le\frac{n}{2}$, then

$$M=\begin{bmatrix} y-1 & 1 & 1\\ y & 1 & 1\\ n-2y & 1 & 0\end{bmatrix},$$
$$M^{-1}=\begin{bmatrix} -1 & 1 & 0\\ n-2y & -n+2y & 1\\ 3y-n & -3y+n+1 & -1\end{bmatrix}.$$

We only need to prove that any pair of vertices (i,j) in D has an $(2y^2-4y+n, 2y^2-3y+n, 2ny-4y^2)-$ walk. We can get the following results,

$$\begin{bmatrix} r\\ s\\ t\end{bmatrix}+(y+r-s)\begin{bmatrix} y-1\\ y\\ n-2y\end{bmatrix}+(ny-2y^2-(n-2y)r+(n-2y)s-t)\begin{bmatrix}1\\1\\1\end{bmatrix}$$
$$+(3y^2-3y-ny+n-(3y-n)r+(3y-n-1)s+t)\begin{bmatrix}1\\1\\0\end{bmatrix}$$
$$=\begin{bmatrix} 2y^2-4y+n\\ 2y^2-3y+n\\ 2ny-4y^2\end{bmatrix}. \qquad (5)$$

Clearly, $r\le y-1$, $s\le y$, $t\le n-2y$, $3y-n>0$, $-3y+n+1<0$. Combined with Fig.1, if $s=y$, then $r\ge 0$, $t\ge 0$; if $r=y-1$, then $s\ge 0$, $t\ge 0$; if $r=y-1$, $t=n-2y$, then $s\ge 0$. Hence $y+r-s\ge 0$, $ny-2y^2-(n-2y)r+(n-2y)s-t\ge 0$ and $3y^2-3y-ny+n-(3y-n)r+(n-3y+1)s+t\ge 0$. By (5), we can see that the walk starts at vertex i, follows P_{ij} to vertex j, goes $y+r-s$ times around the $(n-1)-$cycle, $ny-2y^2-(n-2y)r+(n-2y)s-t$ times around the $3-$ cycle, and $ny+y-3y^2+(n-3y)r-(n-3y+1)s+t$ times around the $2-$cycle is an $(2y^2-4y+n, 2y^2-3y+n,$ $2ny-4y^2)-$walk. So

$$\exp(D) \le 2y^2-4y+n+2y^2-3y+n+2ny-4y^2$$
$$=(2n-7)y+2n.$$

Obviously, $\exp(D)$ is a function of y. Denote $f(y)=(2n-7)y+2n$, then $f'(y)=2n-7$ and $f(y)$ is an increasing function of y. Since $\frac{n+1}{3}<y\le\frac{n}{2}$, we have that

$$\exp(D)\le(2n-7)y+n\le f(\frac{n}{2})=n^2-\frac{3n}{2}.$$

Therefore, by comparing the upper bounds of the primitive exponent in the above four cases, we can get $\exp(D)\le n^2-\frac{3n}{2}$.

Theorem 3 Let D be primitive and $|M|=-x+y=-1$, then

$$\exp(D)\le n^2-\frac{3n}{2}.$$

Proof In this case, the inverse matrix of the cycle matrix M is

$$M^{-1}=\begin{bmatrix} 1 & -1 & 0 \\ -n+2y+2 & n-2y-2 & 1 \\ -3y+n-2 & 3y-n+3 & -1 \end{bmatrix}.$$

Clearly, if $|M|=-x+y=-1$, then $x=y+1$. Associating to the cycle matrix of M, we can see $n-1-x-y=n-1-(y+1)-y=n-2y-2\ge0$, so $y\le\frac{n}{2}-1$, $-n+2y+2\le0$. The size of matrix elements $-3y+n-2$ and $3y-n+3$ are related to the value of y in M^{-1}. The proof process of Theorem 3 is similar to theorem 2. So, we will not go into much detail here.

3. Conclusion

By synthesizing Theorem 2 and Theorem 3, we can get the important conclusion.

Theorem 4 Let D be primitive, then

$$\exp(D)\le n^2-\frac{3n}{2}.$$

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References

[1] Yubin, G.; Yanling S. Exponents of two-colored double directed cycles, *Journal of Heilongjiang University (Natural Science Edition)*, vol.4, no. 2004, pp. 55-58.

[2] B.L., S.; S., S. Exponents of nonnegative matrix pairs, *Linear Algebra Appl*, vol. 363, no. 2003, pp. 275-293.

[3] Yubin, G.; Yanling, S. Exponents of two-colored digraphs with two cycles, *Linear Algebra Appl*, vol. 407, no. 2005, 263-270.

[4] Yanling, S.; Yubin, G.; Liang, S. Exponents of a class of two-colored digraphs, *Linear and Multilinear Algebra*, vol. 53, no. 2005, pp. 175-188.

[5] Qing, Y.; Yanling, S. Exponents of a class of three-colored digraphs, *Journal of North University of China (Natural Science)*, vol. 30:6, no. 2009, pp. 512-517. (In Chinese)

[6] Meijin, L. The primitive sets of a class of two-colored digraphs with one common vertex, *Journal of Jinan University (Natural Sciences and Medicine Edition)*, vol. 34:5, no. 2013, pp. 483-488. (In Chinese)

[7] Meijin, L.; Yubin, G. Primitive exponents of a class of three-colored digraphs with three cycles, *Journal of Shandong University (Natural Science)*, vol. 43:1, n. 2008, pp. 65-72. (In Chinese)

[8] Qing, Y.; Yanling, S. Exponents of a class of three-colored digraphs, *Journal of North University of China (Natural Science)*, vol. 30:6, no. 2009, pp. 512-517. (In Chinese)

[9] Meijin, L.; Xi, Li. Upper bound on of the primitive exponents of a class of three-colored digraphs, *Ars Combinatoria*, vol. 133, no. 2017, pp. 179-195.

[10] Huiling, Z.; Qinyan, S. Exponents of a class of three-colored digraphs, *Journal of Jilin Normal University (Natural Science)*, vol. 5:2, no. 2011, pp. 82-89. (In Chinese)